On quantization of electromagnetic field. I. Classical electrodynamics.

D. A. Arbatsky*

March, 2002

Abstract

A technique for investigation of classical fields is developed on the base of invariant Hamiltonian formalism. Electromagnetic and scalar fields are considered as particular examples of using the general method. Poisson brackets for these fields are calculated. The necessity of introduction of "non-physical" degrees of freedom for electromagnetic field is explained.

1. Introductory remarks. This paper is the first in the set of six joined by the same title. References to these papers will be given here in Roman numbers: [I], ..., [VI].

One of the causes for this research was the attempt to answer the question: What is the structure of the space of states of quantized electromagnetic field, considered from functional analysis?

1. When they describe Gupta-Bleuler quantization scheme modern textbooks silently imply that the question about the topology of quantum space of states of quantized electromagnetic field is solved in full analogy with scalar field. And electromagnetic field is quantized either in usual Hilbert space (such a construction is not relativistic invariant, even in non-apparent way [VI]) or in "Hilbert space with indefinite metric" (in such an approach the question about the topology has not been considered yet with all necessary mathematical strictness; in fact, it turns out that the space of states has not been defined constructively at all).

   Anyway, founded on different analogies, it is usually supposed that the space of states, from *topological* point of view, *must* be a Hilbert space. From our research it will be clear, that this point of view is erroneous. And this result is quite general: it forces us to make a new look at quantization of fields, even those fields that were quantized quite well up to now (for example, the scalar field).

2. It will be shown that the root of the problem is not in functional analysis, but in algebra. The process of quantization in Fock space was not satisfactorily described from *algebraic* point of view yet. This is because, following Fock [1], for construction of quantized electromagnetic field people take as a starting object the one-particle quantum space of states. But such an algebraic structure is too poor: there is no algebraic process that allows to construct quantized field, based on this structure (Namely, using only structure of one-particle state, it is impossible to introduce local field operators (see, for example, [3], chapter 5)).

   It will be clear from this research that an adequate algebraic structure for construction of a quantized field is a classical field, described with the use of the invariant Hamiltonian formalism.

3. After solving algebraic questions (i. e. formulating satisfactory quantization scheme) the solution of the question about topology appears to be a quite "working" (non-fundamental) problem. It appears to be important to realize that it is necessary for constructing the space of states of electromagnetic field to abandon to use Hilbert space. In the paper [VI] for solving this problem we will introduce new types of functional spaces.

4. There is no reason to discuss the question about relativistic invariance of the suggested construction of quantized field. The suggested construction is not only invariant: the requirement of relativistic invariance is the part of the construction.

   If we use the quantization scheme, given here, for quantizing the harmonic oscillator, the requirement of invariance with respect to some group substitutes (in some sense) the requirement of using Hilbert space as

*http://daarb.narod.ru/ , http://wave.front.ru/

a space of states. And we get two quantizations: one in Hilbert space and one — in the space with indefinite scalar product.

So, we do not even try to generalize Stone-von Neumann theorem for the case of relativistic fields: instead, we suggest completely different conception of quantization, which works *differently* even in the case of the harmonic oscillator.

5. So far as our approach to quantization is consistently oriented to explicit control of a symmetry (in the case of relativistic fields — with respect to the Poincare group), it becomes possible to see, how the action of a group of symmetry is transferred from classical space of states to quantum. It will not be an exaggeration to say that these are our papers where it is for the first time explained (at the rigorous level) how a quantum system gets integrals of motion of the same type that corresponding classical system has (and these integrals are connected with the group of symmetry).

6. It is well known from the classical paper of Bohr and Rosenfeld [4] that some problems arise when we consider field operators in a fixed point of space-time. In modern books pretending to mathematical strictness it is said that we have in this case an "operator distribution".

   If we use the invariant Hamiltonian formalism as a base for construction of quantum fields, we can see that this problem can be investigated even in the classical mechanics, and much more clearly.

   From the formal point of view, it is possible to investigate this question even in this paper. Especially as rigorous approach to calculation of Poisson brackets requires discussion of the topology of invariant phase space. Nevertheless, I prefer to postpone this question to the last paper. This is because the choice of topology for invariant phase space can be clearly and simply *motivated* only after consideration relativistic fields from the point of view of group theory [IV] and discussion of quantization [VI].

7. It is known that it is possible to add some divergence to the Lagrangian of a field without changing the equations of the motion. It is shown in the paper [II] that invariant Hamiltonian description of a field is not changed in this case. So far as the suggested scheme of quantization is totally based on the structure of the invariant Hamiltonian formalism, obtained quantization does not depend on this substitution of Lagrangian also.

8. So far as for the base for construction of quantized fields we use classical field, not one-particle quantum space, Wigner-Mackey theory (about unitary representations of the Poincare group) does not play a fundamental role for us. The analogy of this theory is the theory of symplectic representations of Poincare group (in the paper [IV] I tried to describe some foundations).

   The theory of symplectic representations of Poincare group is similar to the theory of unitary representations. But the example of electromagnetic field shows that there is not full analogy here.

9. In our account we will orientate ourself mainly to linear real Bose fields. The requirement of linearity here is of principle: we cannot quantize non-linear fields. As to requirements of reality and being Bose-field, these restriction are accepted here only to simplify notations and formulations. "Complexity" of a field just means the existence of additional complex structure. As to Fermi fields, for their description we need to use Grassmanian variables (see, for example, [6]). It is not important for our consideration, whether we use usual or Grassmanian numbers.

In this paper we give classical description of electromagnetic field. The main goal is to give an account of well-known facts in a new language. However this approach allows us to look at some questions completely differently: for example, the question about energy of electrostatic field was treated incorrectly till now.

2. Notations. In tensor notations we will write vectors of Minkowski space as a_μ , b_ν etc. And we will use only *contravariant* components of tensors; indexes numbering components will be written always *down*. The frame of reference will be always supposed to be orthonormal with respect to the scalar product $g(\,\cdot\,,\,\cdot\,)$, so that metric tensor has standard appearance: $g_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)_{\mu\nu}$. Repeating indexes always imply summation with due regard for signs. For example, the scalar product will be written as

$$a\,b = a_\mu\,b_\mu = g_{\mu\nu}\,a_\mu\,b_\nu = a_0\,b_0 - a_1\,b_1 - a_2\,b_2 - a_3\,b_3 \ .$$

The derivatives with respect to space-time coordinates will be marked by the symbol ∂_μ :

$$\partial_\mu = \left(\ \frac{\partial}{\partial x_0}\ \middle|\ -\frac{\partial}{\partial x_1}\quad -\frac{\partial}{\partial x_2}\quad -\frac{\partial}{\partial x_3}\ \right)_\mu \ ,$$

so, the index of a derivative, like all indexes, is contravariant.

For reducing of notations the symbol of antisymmetrization is introduced: $^{[\mu\nu]}$. This symbol means that we make antisymmetrization with respect to the nearest indexes μ and ν following after it. For example, the tensor of electromagnetic field is written as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = {}^{[\mu\nu]}\, 2\, \partial_\mu A_\nu$.

D'Alembert operator is given by formula: $\partial^2 = \partial_\mu \partial_\mu$.

The Fourier transformation everywhere looks like:

$$\widetilde{\varphi}(k) = \int d^4x\, e^{+ikx}\, \varphi(x) \ , \qquad \varphi(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \widetilde{\varphi}(k) \ . \tag{1}$$

The system of units is so that $\hbar = c = 1$.

3. Invariant Hamiltonian formalism. It was shown in papers [7, 8, 9] that basic notions of the Hamiltonian formalism (namely, phase space and symplectic structure on it) can be treated in a relativistic invariant sense. In those papers formulas for symplectic structure of the most important systems were obtained (in coordinate representation). So, there was given a base of a scheme that can be called an invariant Hamiltonian formalism. A detailed account of this formalism see in original reviews [10, 11]. The question of equivalence of this approach and usual Hamiltonian mechanics for the case of systems with constraints was discussed also in [12].

Nevertheless I will remind here some basic notions. This is necessary in order to introduce our own notations.

Let field $\varphi_i(x)$ is described by the Lagrangian $L(\varphi_i, \partial_\mu \varphi_i)$. Here we will suppose that Lagrangian depends only on field values φ_i and their derivatives[1] and does not depend on the point x . Furthermore, we will consider only linear fields and Lagrangian will be supposed to be a quadratic function.

The least-action principle leads us to Euler-Lagrange equations:

$$\frac{\partial L}{\partial\, \varphi_i} - \partial_\mu \frac{\partial L}{\partial\, (\partial_\mu \varphi_i)} = 0 \ . \tag{2}$$

Invariant phase space Z is defined as the set of all functions[2] φ_i that satisfy equations (2). So long as we suppose the equation (2) to be linear, the space Z has natural linear structure.

The elements of the space Z we will denote by underlined sets of symbols; for example, $\underline{c} \in Z$. Functions on this space we will write as $f^{\underline{c}}$. For example, the value of the function φ corresponding to an element $\underline{c}$ in point x is written as $\varphi(x)^{\underline{c}}$.

We will usually identify elements of tangent bundle TZ with elements of the space Z . If it is necessary to point out that tangent vector $\underline{c}$ comes from a point $\underline{b} \in Z$, we can write it as $\underline{c}[\underline{b}]$. Elements of cotangent bundle T^*Z will be similarly identified with the elements of adjoint space Z^* . At the same time, differentials of linear functions will be identified with functions, i. e. instead of writing df we write simply: f .

On the invariant phase space Z , like on usual phase space, there is a symplectic structure ω :

$$\omega = \int\limits_\Sigma d\sigma_\mu\, \delta j_\mu(x) \ , \qquad \text{here} \quad \delta j_\mu(x) = \frac{\partial L}{\partial\, (\partial_\mu \varphi_i)} \wedge \varphi_i \ . \tag{3}$$

The integration is assumed over any space-like hyper-surface Σ , which behaves well enough in infinity. 2-form δj_μ is called a symplectic current. So long as the symplectic current is conserved, the result does not depend on the choice of the surface Σ .

The freedom in the choice of Lagrangian and non-uniqueness of the symplectic structure will be considered in the paper [II]. In reasonable sense we can say that non-uniqueness in the choice of Lagrangian does not influence the symplectic structure.

The symplectic structure ω , as usually [13], defines an isomorphism $I : T^*Z \to TZ$ of cotangent T^*Z and

[1] It will be sufficient here to restrict ourself with Lagrangians depending only on derivatives of the first order. In general case we can consider also Lagrangians with higher derivatives [12].

[2] More precisely, we will suppose for the present that we consider only smooth functions. We will also suppose that these functions differ from zero only on such a set, that intersection of this set with any space-like plane is finite. In paper [VI] we will discuss the question about natural topology of the space Z .

tangent TZ bundles[3]. It transfers any 1-form $l^{\cdot} \in T^*Z$ to tangent vector $\underline{Il} \in TZ$, so that for any tangent vector $\underline{c} \in TZ$ we have the equality:

$$\omega^{\underline{c}\,;\underline{Il}} = l^{\underline{c}} \ . \tag{4}$$

If a form $l^{\cdot}$ is a differential of some function $g^{\cdot}$, i. e. $l^{\cdot} = dg^{\cdot}$, we say that $g^{\cdot}$ is a generator of the vector field $\underline{Il} = \underline{I\,dg}$.

The Poisson bracket of two functions $f^{\cdot}$ and $g^{\cdot}$ is defined by equality:

$$\{\, f\,,\, g\,\} = df^{\underline{I\,dg}} \ . \tag{5}$$

If functions $f^{\cdot}$ and $g^{\cdot}$ are linear this definition can be written just as:

$$\{\, f\,,\, g\,\} = f^{\underline{Ig}} \ . \tag{6}$$

It will be enough here to restrict ourself with real fields, so the space Z will be a real linear space. Nevertheless, it is useful to consider complex functions on the phase space. Their differentials belong to complexified adjoint space. This is why instead of using the space Z^* we will always use its complexification $Z^*_{\mathbb{C}}$. Furthermore, when we define the Poisson bracket by formula (5), it is necessary to consider complex vectors $\underline{I\,dg}$. It could be avoided if we define the Poisson bracket in a more abstract way[4].

4. Scalar field. Consider the scalar field. Its Lagrangian is:

$$L = \tfrac{1}{2}\,\partial_\nu\varphi\,\partial_\nu\varphi - \tfrac{m^2}{2}\,\varphi^2 \ .$$

The equation of the motion (2) takes form (Klein-Fock-Gordon equation):

$$\left(\partial^2 + m^2\right)\varphi = 0 \ . \tag{7}$$

The symplectic structure (3) on the invariant phase space Z^{scal} in coordinate representation takes the form (given in [7]):

$$\omega = \int\limits_{\Sigma} d\sigma_\nu\,\partial_\nu\varphi \wedge \varphi \ . \tag{8}$$

Let us make a Fourier transformation of the function $\varphi(x)$ by the formulas (1). So far as the function $\varphi(x)$ satisfies the equation (7), its Fourier-transform $\widetilde{\varphi}(k)$ can be represented as:

$$\widetilde{\varphi}(k) = 2\pi\,\delta(k^2 - m^2)\cdot a(k) \ , \tag{9}$$

where $a(k)$ is a usual function defined on the mass surface $k^2 = m^2$.

Performing Fourier transformation of the symplectic structure (8) we have:

$$\omega^{\cdot\,;\cdot\,\cdot} = \int d\mu_m \cdot i\,\varepsilon(k)\cdot a(-k)^{\cdot}\,a(k)^{\cdot} \ , \qquad \text{here } \ d\mu_m = \frac{d^4k}{(2\pi)^4}\cdot 2\pi\,\delta(k^2 - m^2) \ . \tag{10}$$

Here integrated function is a usual product of numbers[5]. It is implied that arguments in the right part follow in the same order as in the left.

Now in order to calculate the Poisson bracket of two field values $\{\,\widetilde{\varphi}(k)\,,\,\widetilde{\varphi}(k')\,\}$, according to the formula (6), let us consider the vector $\underline{I\widetilde{\varphi}(k')}$. From the formulas (4) and (10) we have that for any vector $\underline{c}$:

$$\int d\mu_m \cdot i\,\varepsilon(k)\cdot a(-k)^{\underline{c}}\,a(k)^{\underline{I\widetilde{\varphi}(k')}} = \widetilde{\varphi}(k')^{\underline{c}} \ .$$

---

[3] Rigorously speaking, we can talk about such an isomorphism only after discussing the topology of the invariant phase space. Without such a discussion even the notion of cotangent bundle does not have a clear sense. For the reasons explained in "Introductory remarks" discussion of this question for the case of relativistic fields we postpone to paper [VI].

It should be also kept in mind that values like $\varphi(x)$ that we use here appear to be not belonging to cotangent bundle after we introduce the proper topology. This shortcoming can be easily eliminated, but prefer just to ignore it till discussion of the topology.

[4] We prefer to avoid complexification of Z, because this space can be considered as the set of *physical* states existing in nature. The space $Z^*_{\mathbb{C}}$ can be considered as the set of *mathematical* values that we use to describe physical states.

[5] If we denote $d\mu^+_m = d\mu_m\cdot\theta(k)$, the formula (10) can be written with external product of forms:

$$\omega = \int d\mu^+_m \cdot i\,a^*(k)\wedge a(k) \ .$$

Using linear independence of the form $a(k)\dot{-}$ with different values of k, $k^2 = m^2$, and the definition of this form (9), we get that the vector $\underline{I\widetilde{\varphi}(k')}$ satisfies the following equation:

$$a(k)\,\underline{I\widetilde{\varphi}(k')} = -\,i\,\varepsilon(k)\cdot(2\pi)^4\delta(k+k')\ .$$

Therefore, using the formula (6), for the Poisson bracket we have:

$$\{\,\widetilde{\varphi}(k)\,,\,\widetilde{\varphi}(k')\,\} = -\Big[\,i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2-m^2)\,\Big]\cdot(2\pi)^4\delta(k+k')\ . \tag{11}$$

Sometimes it is useful to have an analogous relation in coordinate representation. In order to get it we make the Fourier transformation of the formula (11) with respect to the arguments k and k':

$$\{\,\varphi(x)\,,\,\varphi(x')\,\} = -D_m(x-x')\ .$$

Here function $D_m(y)$ is:

$$D_m(y) = \int\frac{d^4k}{(2\pi)^4}\,e^{-iky}\cdot\Big[\,i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2-m^2)\,\Big]\ . \tag{12}$$

5. "Physical" electromagnetic field. Let us choose the Lagrangian of electromagnetic field gauge-invariant:

$$L = -\,\tfrac{1}{4}\,F_{\mu\nu}\,F_{\mu\nu} = -\,{}^{[\mu\nu]}\,\partial_\mu A_\nu\,\partial_\mu A_\nu\ ,\qquad \text{here}\quad F_{\mu\nu} = {}^{[\mu\nu]}\,2\,\partial_\mu A_\nu\ . \tag{13}$$

The equations of the motion (2) take the form (the second Maxwell equation):

$$\partial_\mu F_{\mu\nu} = 0\ . \tag{14}$$

In this case it is convenient to define the invariant phase space $Z^{\rm phys}$ as the set of functions $F_{\mu\nu}(x)$ that satisfy the equation (14). The value of vector potential A_μ in a fixed point x is not a well-defined function on $Z^{\rm phys}$: an expression like $A_\mu(x)\dot{-}$ has no sense if we do not say which gauge we use. For this reason, it is impossible to give any one-valued formula for the Poisson bracket of vector potential.

The symplectic structure (3) in this case takes the form (shown in [8, 10]):

$$\omega = -\int\limits_\Sigma d\sigma_\mu\,F_{\mu\nu}\wedge A_\nu\ . \tag{15}$$

It is appropriate to mention that, though the form ω is written with using gauge-dependent value A_ν, it is still gauge-independent, of course.

Let us now choose the vector potential A_μ so that it satisfies the Lorentz condition $\partial_\nu A_\nu = 0$. According to the equation (14), it will also satisfy the D'Alembert equation: $\partial^2 A_\mu = 0$. Let us split the expression (15) into two terms:

$$\omega = -\int\limits_\Sigma d\sigma_\mu\,\partial_\mu A_\nu\wedge A_\nu + \int\limits_\Sigma d\sigma_\mu\,\partial_\nu A_\mu\wedge A_\nu\ .$$

The second integral is equal to zero. We can check it in the following way. First, from the Lorentz condition and antisymmetry of external product we have that the integrated function is a conserved current: $\partial_\mu(\partial_\nu A_\mu\wedge A_\nu) = 0$. Therefore in the second integral, independently from the first, we can integrate on the surface $t = 0$ instead of integration on Σ. So, we get the integral:

$$\int d^3\boldsymbol{x}\,\partial_\nu A_0\wedge A_\nu\ .$$

Using the Lorentz condition we can write it as:

$$\int d^3\boldsymbol{x}\,\partial_\nu(A_0\wedge A_\nu)\ .$$

But $A_0\wedge A_0 = 0$. Therefore the summation by index ν can be performed just from 1 to 3. We get:

$$\int d^3\boldsymbol{x}\,\partial_n(A_0\wedge A_n)\ .$$

This integral is an integral of the three-dimensional vector $A_0 \wedge A_n$. By Ostrogradsky-Gauss divergence theorem, it is equal to the flux of this vector through a far two-dimensional closed surface. This flux is equal to zero because the vector potential is implied to be chosen so that it is equal to zero there.

So, under the Lorentz condition $\partial_\nu A_\nu = 0$, the symplectic structure (15) can be written in the following way:

$$\omega = -\int\limits_{\Sigma} d\sigma_\mu \, \partial_\mu A_\nu \wedge A_\nu \ . \tag{16}$$

Let us make the Fourier transformation (1) of functions $A_\mu(x)$ and $F_{\mu\nu}(x)$. Similarly to the case of the scalar field we can write these Fourier transforms as:

$$\widetilde{A}_\mu(k) = 2\pi\,\delta(k^2)\cdot a_\mu(k) \ , \qquad \widetilde{F}_{\mu\nu}(k) = 2\pi\,\delta(k^2)\cdot f_{\mu\nu}(k) \ .$$

Here $a_\mu(k)$ and $f_{\mu\nu}(k)$ are usual functions defined on the light cone $k^2 = 0$.

The symplectic structure (16) in the Fourier representation takes the form[6]:

$$\omega^{\vdots\,\vdots\,\vdots} = -\int d\mu_m \cdot i\,\varepsilon(k)\cdot a_\nu(-k)^{\vdots}\, a_\nu(k)^{\vdots} \ . \tag{17}$$

The formulas (16) and (17) look very similar to the corresponding formulas (8) and (10) for the scalar field. More rigorously, they look like formulas for the symplectic structure of four scalar fields. But it is important to understand that the invariant phase space $Z^{\rm phys}$ introduced in this section is not a direct sum of phase spaces of four scalar fields. For this reason, as we mentioned above, it is impossible to calculate the Poisson bracket for vector potential.

Consider now the structure of the space $Z^{\rm phys}$ in Fourier representation. This space can be considered as the set of functions $a_\mu(k)$ on the light cone $k^2 = 0$ satisfying the Lorentz condition $-\,i\,k_\mu\,a_\mu(k) = 0$ and considered accurate up to the gauge transformation:

$$a_\mu(k) \ \sim \ a_\mu(k) - i\,k_\mu\,\lambda(k) \ .$$

From here and from the formula (17) we see that the symplectic structure on the space $Z^{\rm phys}$ is not degenerate.

Now it is clear that, though it is impossible to calculate the Poisson brackets for components of vector potential, it is possible to calculate the brackets $\{\,\widetilde{F}_{\mu\rho}(k)\,,\,\widetilde{F}_{\nu\sigma}(k')\,\}$ and $\{\,F_{\mu\rho}(x)\,,\,F_{\nu\sigma}(x')\,\}$. Nevertheless, we postpone this problem till the section 11.

6. Heisenberg-Pauli term. After the paper of Heisenberg and Pauli [14] when quantum theory of electromagnetic field is discussed it is customary to change the Lagrangian adding to it an additional term $\frac{\varepsilon}{2}\,(\partial_\mu A_\mu)^2$. For such a step in modern textbooks there are many different "explanations". Generally they can be divided into two types.

The first type are "explanations" appealing to formal difficulties of using the Hamiltonian formalism and watching relativistic invariance. As we have shown in the section 5, even if there were any difficulties, they disappear, if we use the invariant Hamiltonian formalism.

The second type are "explanations" appealing to quantum theory in some way. Now our first goal is to explain that the reasons for changing the Lagrangian exist even in the classical theory.

At first sight, the term $\frac{\varepsilon}{2}\,(\partial_\mu A_\mu)^2$ only fixes the gauge: if the Lorentz condition was satisfied in the past, it will be satisfied in the future. But actually there is a more deep consequence: when we develop scattering theory and perturbation theory we get the opportunity to get rid of such a fiction as separation of field to "own" field of particles and "radiation".

Indeed, it is impossible in practice to solve a dynamic problem exactly. So, we calculate first some first approximation and then we calculate radiative corrections. In order to take into account radiative recoil we have to divide field into "own" and "radiative" in some way. But it is impossible to do it rigorously even in simplest situations. It is not surprising, because in the Lagrangian formalism there is no hint of such a separation.

[6] Using an external product we can write it as:

$$\omega = -\int d\mu_m^+ \, i\,a_\nu^*(k) \wedge a_\nu(k) \ .$$

But we can approach problems of scattering differently. Consider, for example, scattering of two particles interacting with each other by electromagnetic field. Let us use some naive mechanical analogy: for example, two heavy balls sliding on a smooth elastic membrane. Membrane corresponds to electromagnetic field.

If we do not divide the field into own and radiation we have to accept that all field is "created" by particles. In order to formalize such an approach we have to introduce turning interaction on and off.

The process of scattering of particles and of electromagnetic field then can be described in the following way:

1. In the far past the particles are free and they do not interact with the field. There exists also the field and it is far from the particles.
2. Then, while particles are far from each other, we turn on interaction adiabatically. The particles put on their field at this stage.
3. After interaction is fully turned on particles (and field) run to each other near, interact and then run away from each other.
4. When they are far enough we turn interaction off adiabatically.
5. Then there remain only bare particles, and there is no field near them, and free field which is not connected with the particles.

It is obvious for balls sliding on an elastic membrane how to turn on and off interaction: it is necessary to turn on and off the gravitation. But in the case of electromagnetic field we can not do anything so simple. Consider, for example, interaction of electromagnetic field $A_\mu(x)$ with a given current $J_\mu(x)$. The Lagrangian in this case takes the form:

$$L = -\tfrac{1}{4}\, F_{\mu\nu}\, F_{\mu\nu} - \alpha\, J_\mu\, A_\mu \ . \tag{18}$$

Here $\alpha(x)$ is a function that turns on and off interaction. In order to write formulas shorter we will use a notation:

$$J^\alpha_\mu(x) = \alpha(x)\, J_\mu(x) \ .$$

From the Lagrangian (18) we get the equation of the motion:

$$\partial_\mu F_{\mu\nu} = J^\alpha_\nu \ . \tag{19}$$

Applying to both sides the operation ∂_ν we see that, if $\partial_\nu J^\alpha_\nu \neq 0$, the equation (19) does not have any solutions. So, the interaction can not be turned on and off.

But if we add to the Lagrangian the term $\frac{\varepsilon}{2}\,(\partial_\mu A_\mu)^2$, i. e. use the Lagrangian

$$L = -\tfrac{1}{4}\, F_{\mu\nu}\, F_{\mu\nu} + \tfrac{\varepsilon}{2}\,(\partial_\mu A_\mu)^2 - J^\alpha_\mu\, A_\mu \ ,$$

then the equation of the motion takes the form:

$$\partial_\mu F_{\mu\nu} - \varepsilon\, \partial_\nu \partial_\mu A_\mu = J^\alpha_\nu \ , \tag{20}$$

and solution $A_\mu(x)$ can be easily found for any function $\alpha(x)$.

Later we will always imply that $\varepsilon = -1$. In such a case the equation (20) takes especially simple form:

$$\partial^2 A_\nu = J^\alpha_\nu \ . \tag{21}$$

Its solution is the Liénard-Wiechert potential. For this reason this "gauge" should be called the Liénard-Wiechert gauge. Nevertheless, in quantum electrodynamics it is called the Feynman gauge.

If we calculate a four-dimensional divergence of both parts of equation (21), we get:

$$\partial^2(\partial_\nu A_\nu) = \partial_\nu J^\alpha_\nu \ .$$

It can be shown from here that, if interaction is turned on and off adiabatically and the Lorentz condition $\partial_\nu A_\nu = 0$ took place in the far past, then it will take place everywhere.

From this we also see that the additional term $\frac{1}{2}\,(\partial_\mu A_\mu)^2$ in the Lagrangian will make a zero contribution in the variation of action. Therefore in the region, where interaction is completely turned on, the system of particles and fields will satisfy the same equations of motion as the system with no additional term in the Lagrangian.

So, introduction of the additional term in the Lagrangian does not substantially change the scattering. But it allows to avoid separation of "own" field of particles and allows to construct a local perturbation theory.

It is difficult to say now, how far we can go in constructing of perturbation theory for classical electrodynamics. But it is apparent that without introduction of the additional term in Lagrangian situation would be completely hopeless.

7. "Non-physical" electromagnetic field. Now consider again the free electromagnetic field, but include already in Lagrangian the term $-\frac{1}{2}(\partial_\mu A_\mu)^2$. Then the Lagrangian can be written in the following way:

$$L = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu A_\mu)^2 = -\tfrac{1}{2}\,\partial_\mu A_\nu\,\partial_\mu A_\nu + \tfrac{1}{2}\,\partial_\mu(A_\nu\,\partial_\nu A_\mu - A_\mu\,\partial_\nu A_\nu) \ .$$

As we will show in the paper [II] the term that is a full divergence can be thrown off. After that the Lagrangian takes especially simple form:

$$L = -\tfrac{1}{2}\,\partial_\mu A_\nu\,\partial_\mu A_\nu \ ,$$

i. e. it is in fact the Lagrangian of four scalar fields[7] with $m = 0$ considered in the section 4. The equation of the motion has the form:

$$\partial^2 A_\mu = 0 \ .$$

Now the invariant phase space is a direct sum of four phase spaces of scalar fields. Let us denote it as Z^4 . The symplectic structure in coordinate and Fourier representations is given by formulas:

$$\omega = -\int\limits_\Sigma d\sigma_\mu\,\partial_\mu A_\nu \wedge A_\nu \ . \tag{22}$$

$$\omega^{\vdots\,\vdots\,\vdots} = -\int d\mu_m \cdot i\,\varepsilon(k)\cdot a_\nu(-k)^{\vdots}\,a_\nu(k)^{\vdots} \ . \tag{23}$$

The formulas (22) and (23) look the same as corresponding formulas from the section 5. But the difference in their meaning is important: now we can calculate the Poisson bracket for the vector potential. In Fourier representation it has the form:

$$\{\,\widetilde{A}_\mu(k)\,,\,\widetilde{A}_\nu(k')\,\} = g_{\mu\nu}\cdot\Big[\,i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2)\Big]\cdot(2\pi)^4\delta(k+k') \ . \tag{24}$$

In coordinate representation:

$$\{\,A_\mu(x)\,,\,A_\nu(x')\,\} = g_{\mu\nu}\,D_0(x-x') \ . \tag{25}$$

Here $D_0(y)$ is the function (12) with $m = 0$. In this case it can be written just as:

$$D_0(y) = \frac{1}{2\pi}\,\varepsilon(y)\,\delta(y^2) \ .$$

Let us apply to both sides of the formula (24) symbol ${}^{[\rho\mu]}\,2\,(-i\,k_\rho)\,{}^{[\sigma\nu]}\,2\,(-i\,k'_\sigma)$. We get the formula for the Poisson bracket of the tensor of electromagnetic field in the Fourier representation:

$$\{\,\widetilde{F}_{\mu\rho}(k)\,,\,\widetilde{F}_{\nu\sigma}(k')\,\} = {}^{[\mu\rho]\,[\nu\sigma]}\,4\,g_{\mu\nu}\,k_\rho\,k_\sigma\cdot\Big[\,i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2)\Big]\cdot(2\pi)^4\delta(k+k') \ . \tag{26}$$

In the coordinate representation this relation can be written in the following way:

$$\{\,F_{\mu\rho}(x)\,,\,F_{\nu\sigma}(x')\,\} = -\,{}^{[\mu\rho]\,[\nu\sigma]}\,4\,g_{\mu\nu}\,\partial_\rho\,\partial_\sigma D_0(x-x') \ . \tag{27}$$

Here both derivatives are taken with respect to the argument without dash.

It should be mentioned already that formulas (26) and (27) are the same as formulas for the Poisson brackets of the "physical" electromagnetic field described in the section 5. It will be proven in the section 11.

8. Energy of electrostatic field. In the paper [III] we will get a formula that can be used in the invariant Hamiltonian formalism for obtaining generators of the Poincare group. With its help we easily get the formula for the momentum of the "non-physical" electromagnetic field:

$$P_\nu = -\int d\mu_m^+ \cdot k_\nu\,a_\rho^*(k)\,a_\rho(k) \ . \tag{28}$$

From this formula we see that the energy of the "non-physical" field is not positive-definite. And what is more, it follows from this formula that the energy stored in electrostatic field created by a system of fixed charges is always *negative*.

[7] The term corresponding to the field with index $\nu = 0$ is included with negative sign.

Here we will discuss this paradox with some details. This is necessary because it was usually incorrectly treated in the frame of *quantum* theory (see, for example, [16, 17, 18, 19]).

The problem of negativity of energy of "time" photons in quantum electrodynamics was noted a long time ago. It was supposed that quantization in the space with indefinite scalar product allows change the sign of this energy. And it was considered as an argument for introduction of indefinite scalar product. It seemed so because, if we use indefinite scalar product in quantum case, it is not quite clear what we should call the energy of a state: the average value of the Hamiltonian or its eigenvalue. This vagueness, of course, appeared because, first, there was no clear conception of quantization, and second, classical theory was not formulated in the appropriate way.

But from our consideration it is clear that the discussed energy *must* be negative: only such value looks natural from the point of view of relativistic theory. And indefiniteness of metric in quantum case has no relation to this question.

It should be also noted here that the notion of vacuum of a field we do not connect in any way with minimum of energy. The vacuum of a classical field is just the zero vector of the space Z . A notion of vacuum of a quantum field we will introduce in description of quantization in the paper [VI].

But what can we say now about the positiveness of electrostatic energy which is well-known in the usual electrostatics? The fact is that this notion implies usually a different meaning: it usually implies the work that the system can make, if charges will be split in infinite small parts and they will be carried far away from each other. But the formula (28) gives the energy that remains in the field, if its interaction with the charges is instantly turned off.

We will not discuss this question in more details here. But let us notice the following. In electrostatics there is the formula for energy:

$$E^{\,\mathrm{full}} = \tfrac{1}{2} \int \varphi \, dQ \ .$$

The multiplier $\frac{1}{2}$ is usually explained so that, if we use the formula for the energy in external field, we count the mutual energy of two charges two times.

But in the frame of the theory that we discuss here the field is always considered as external. Therefore, we can say that the energy of charges is always given by the formula:

$$E^{\,\mathrm{charges}} = \int \varphi \, dQ \ .$$

We have to add to it the energy of the field:

$$E^{\,\mathrm{field}} = \, - \tfrac{1}{2} \int \varphi \, dQ \ .$$

The sum of $E^{\,\mathrm{charges}}$ and $E^{\,\mathrm{field}}$ is exactly $E^{\,\mathrm{full}}$.

9. Scattered states of "non-physical" field. Consider now scattering of the "non-physical" field on a given current. As it was explained in the section 6, we will suppose that the interaction with electromagnetic field is turned on and off adiabatically and the particles constituting the current are accelerated only in the region of space-time where the interaction is fully turned on, i. e. $\alpha(y) = 1$.

In an arbitrary point of space-time the field will consist of two components: the field that existed in the far past and the field created by the particles. This additional field can be calculated with the Liénard-Wiechert formula:

$$A_\mu^{\mathrm{ret}}(x) = \int d^4x' \, D_0^{\mathrm{c}}(x - x') \, J_\mu^\alpha(x') \ , \qquad \text{here} \ \ D_0^{\mathrm{c}}(y) = \theta(y) \, D_0(y) \ . \tag{29}$$

In the invariant Hamiltonian formalism a state of a field implies a solution of the corresponding homogeneous equation. In our case this is the equation $\partial^2 A_\mu = 0$. Therefore when we talk about a state of the field in the far past or in the far future it is natural to continue the function $A_\mu(x)$ by the homogeneous equation to the whole space. So, we come to the definition of the in- and out- fields: $A_\mu(x)^{\underline{\mathrm{in}}}$ and $A_\mu(x)^{\underline{\mathrm{out}}}$. Here $\underline{\mathrm{in}}$ and $\underline{\mathrm{out}}$ are vectors of the invariant phase space Z^4 . It is natural to call these vectors in- and out- states. The difference of these two vectors corresponds to the radiated field:

$$\underline{\mathrm{out}} - \underline{\mathrm{in}} = \underline{\mathrm{rad}} \ .$$

In order to find the field $A_\mu(x)^{\underline{\mathrm{rad}}}$ we have just to make a minor correction in the Liénard-Wiechert formula (29):

$$A_\mu(x)^{\underline{\mathrm{rad}}} = \int d^4x' \, D_0(x - x') \, J_\mu^\alpha(x') \ .$$

In the Fourier-representation this formula can be written in an especially simple way:

$$\widetilde{A}_\mu(k)^{\underline{\text{rad}}} = i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2)\cdot \widetilde{J}^\alpha_\mu(k)\ ,$$

or, with using non-singular function $a_\mu(k)$ on the light cone, even shorter:

$$a_\mu(k)^{\underline{\text{rad}}} = i\,\varepsilon(k)\,\widetilde{J}^\alpha_\mu(k)\ . \tag{30}$$

We see that in the given formulas we can approach to the adiabatic limit. The set of all possible vectors $\underline{\text{rad}}$ forms a linear subspace of the invariant phase space Z^4. Let us denote this space $Z^\perp$. As it was explained in the section 6, if the interaction is turned on and off adiabatically, the Lorentz condition is not broken. Therefore the space $Z^\perp$ does not coincide with the whole Z^4. On the other hand, it follows from the formula (30) that there is no other condition that restricts the space $Z^\perp$. So, $Z^\perp$ is the subspace of the space Z^4 defined by the Lorentz condition:

$$\underline{\text{rad}} \in Z^\perp \quad\Longleftrightarrow\quad -i\,k_\mu\,a_\mu(k)^{\underline{\text{rad}}} = 0\ . \tag{31}$$

This explains the notation $Z^\perp$.

It is natural to believe that all possible in-states were created from the vacuum by interaction with currents. Then vectors of in- and out- fields also lie in the subspace $Z^\perp$. So far as all possible radiation fields fill the whole $Z^\perp$, all possible in- and out- fields also fill the whole $Z^\perp$.

Using the condition (31) and the formula (28) for the energy-momentum vector we see that the states from the space $Z^\perp$ have non-negative energy.

10. Remainders of gauge invariance. Consider now such a vector $\underline{\text{gauge}} \in Z$ that the function $a_\mu(k)^{\underline{\text{gauge}}}$ can be written as:

$$a_\mu(k)^{\underline{\text{gauge}}} = -\,i\,k_\mu\,\lambda(k)\ ,$$

where $\lambda(k)$ is some function on the light cone. The set of all vectors of this type form a linear subspace in Z^4. Let us denote this space $Z^\parallel$. It is obvious that $Z^\parallel$ lies in $Z^\perp$.

Consider again the scattering of particles and field when the motion of particles is not given and there is, generally, non-zero in-field. This scattering is described by the Lagrangian:

$$L = L^{\text{particles}} - \tfrac{1}{2}\,\partial_\mu A_\nu\,\partial_\mu A_\nu - \alpha\,J_\mu\,A_\mu\ .$$

Let us suppose that the corresponding equations of motion are solvable and trajectories of particles and field $A_\mu(x)$ that make action stationary are found. Let us also suppose that adiabatic turning on (off) of interaction is made in so far past (future) that all particles are far from each other and the field $A_\mu(k)^{\underline{\text{in}}}$ ($A_\mu(k)^{\underline{\text{out}}}$) is far from all particles and the field $A_\mu(k)^{\underline{\text{gauge}}}$ is also far from all particles.

Let us add to the field $A_\mu(x)$ the field $A_\mu(x)^{\underline{\text{gauge}}}$. Or, in other words, let us make the gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\ , \tag{32}$$

where $\Lambda(x)$ is a function whose Fourier transform is connected with $\lambda(k)$ by relation:

$$\widetilde{\Lambda}(k) = 2\pi\,\delta(k^2)\cdot\lambda(k)\ .$$

In the region where interaction is turned on and off this addition can not influence the stationarity of action. Indeed, so far as the particles are outside of the region where this addition is present, variations of terms $L^{\text{particles}}$ and $-\alpha\,J_\mu\,A_\mu$ can not change. Furthermore, the term $-\frac{1}{2}\,\partial_\mu A_\nu\,\partial_\mu A_\nu$ differs from the gauge-invariant by the full divergence, which can not change the variation, and by the term $-\frac{1}{2}\,(\partial_\mu A_\mu)^2$, which has a zero variation, because $\partial_\mu A_\mu = 0$.

Consider now the region, where interaction is fully turned on. The transformation (32) can not change the variations of the terms $L^{\text{particles}}$ and $-\frac{1}{2}\,\partial_\mu A_\nu\,\partial_\mu A_\nu$ for the same reasons that in the case of the regions where the interaction is turned on and off. There remains the term $-J_\mu\,A_\mu$. Under the gauge transformation (32) it becomes $-J_\mu\,A_\mu - J_\mu\,\partial_\mu\Lambda(x)$. Therefore, in the variation of the action we get the additional term $-\delta J_\mu\,\partial_\mu\Lambda(x)$. Integrating it by parts and using conservation of the current we get that it does not add anything to the variation.

So, under the transformation (32) motions that satisfy the condition of stationarity of action are transformed to the motions that also satisfy this condition. It is possible to say that the space $Z^\parallel$ does not participate in scattering at all.

It is natural to suppose that the scattered states of electromagnetic field are observed only by their influence on charged particles. Then any two vectors from $Z^{\perp}$ that differ by vector from $Z^{\parallel}$ are physically indistinguishable. Therefore, instead of using the space $Z^{\perp}$, we can use the factor-space $Z^{\perp/\parallel} = Z^{\perp}/Z^{\parallel}$.

Notice now that in the Fourier representation the space $Z^{\perp/\parallel}$ can be considered as the set of functions $a_\mu(k)$ on the light cone $k^2 = 0$ satisfying the Lorentz condition $-i\,k_\mu\,a_\mu(k) = 0$ and considered accurate up to the gauge transformation:

$$a_\mu(k) \;\sim\; a_\mu(k) - i\,k_\mu\,\lambda(k)\;.$$

Therefore, we can establish the natural one-to-one correspondence between elements of the spaces $Z^{\perp/\parallel}$ and Z^{phys}.

And what is more, the space $Z^{\perp/\parallel}$ naturally inherits from the full space Z^4 the symplectic structure (23), and this structure obviously is the same as the symplectic structure of the space Z^{phys} given by the formula (17). So, $Z^{\perp/\parallel}$ and Z^{phys} are also naturally identified as *symplectic* spaces.

11. About the Poisson bracket on $Z^{\perp/\parallel}$. Consider now on the space Z^4 a linear function $\widetilde{F}_{\mu\rho}(k)\dot{-}$ (or $F_{\mu\rho}(x)\dot{-}$). And let us suppose k (or, correspondingly, x) to be fixed. As we have said in the section 3, this function is a generator of a vector field on Z^4.

Notice, first, that so far as the function under consideration is linear on Z^4, its differential does not depend on the point of the space Z^4 (and is identical with the function). Therefore the obtained vector field is constant on the whole space Z^4.

Second, differential of the function under consideration, like the function, is zero on $Z^{\parallel}$. Therefore, as it is seen from the formula for symplectic structure (23), the vector of the obtained constant vector field belongs to the subspace $Z^{\perp}$.

From these two statements we get that the obtained field can be naturally restricted to the factor-space $Z^{\perp/\parallel}$.

On the other hand, the space $Z^{\perp/\parallel}$ has its own symplectic structure. If we restrict the function $\widetilde{F}_{\mu\rho}(k)\dot{-}$ (or $F_{\mu\rho}(x)\dot{-}$) to this factor-space, the obtained function will be a generator of some vector field in $Z^{\perp/\parallel}$.

But, so far as the symplectic structure in $Z^{\perp/\parallel}$ is inherited from Z^4 we can see that both vector fields in $Z^{\perp/\parallel}$ are identical.

From here we also get that the formulas (26) and (27) obtained in the section 7 for the "non-physical" field work for the "physical" field also.

References

[1] V. A. Fock "Konfigurationsraum und zweite Quantelung", Z. Phys. **75**, 622-647 (1932). [in Russian: „Konfiguratsionnoe prostranstvo i vtorichnoe kvantovanie", in [2], pages 25-51.]

[2] V. A. Fock *„Raboty po kvantovoy teorii polya"*, L.: izd-vo LGU (1957).

[3] S. Weinberg *"The quantum theory of fields"*, v. 1: *"Foundations"*, Cambridge univ. press (1996).

[4] N. Bohr, L. Rosenfeld "Zur Frage der Messbarkeit der elektromagnetischen Feldgrössen", Kgl. Danske Vidensk. Selskab., Math.-Fys. Medd. **12**(8), 3-65 (1933). [in Russian: „K voprosu ob izmerimosti elektromagnitnogo polya", in: [5], v. 2, pages 120-162.]

[5] N. Bohr *„Izbrannye nauchnye trudy"*, v. 2, M.: Nauka (1971).

[6] V. I. Ogievetski, L. Mezinchesku „Simmetriya mezhdu bozonami i fermionami i superpolya", UFN **117**(4), 637-683 (1975).

[7] E. Witten "Interacting field theory of open superstrings", Nucl. Phys. **B276**, 291-324 (1986).

[8] Č. Crnković "Symplectic geometry and (super-) Poincaré algebra in geometrical theories", Nucl. Phys. **B288**, 419-430 (1987).

[9] G. J. Zuckerman "Action principles and global geometry", in *Mathematical aspects of string theory*, ed. S. T. Yau, Singapore: World scientific, 259-284 (1987).

[10] Č. Crnković, E. Witten "Covariant description of canonical formalism in geometrical theories", in *Three hundred years of gravitation*, eds. S. W. Hawking, W. Israel, Cambridge: Cambridge univ. press, 676-684 (1987).

[11] Č. Crnković "Symplectic geometry of the covariant phase-space", Class. Quant. Grav. **5**(12), 1557-1575 (1988).

[12] G. Barnich, M. Henneaux, C. Schomblond "Covariant description of the canonical formalism", Phys. Rev. D **44**(4), R939-R941 (1991).

[13] V. I. Arnold *„Matematicheskie metody klassicheskoy mekhaniki"*, M.: Nauka (1989).

[14] W. Heisenberg, W. Pauli "Zur Quantendynamik der Wellenfelder", Z. Phys. **56**(1/2), 1-61 (1929). [in Russian: „K kvantovoy dinamike volnovykh poley", in [15], pages 30-88.]

[15] V. Pauli *„Trudy po kvantovoy teorii"*, M.: Nauka, v. 2 (1977).

[16] S. N. Gupta "Theory of longitudinal photons in quantum electrodynamics", Proc. Phys. Soc. A **63**(7), 681-691 (1950).

[17] S. N. Gupta "Quantum mechanics with an indefinite metric", Can. J. Phys. **35**(8), 961-968 (1957).

[18] K. Nad' *„Prostranstva sostoyaniy s indefinitnoy metrikoy v kvantovoy teorii polya"*, M.: Mir (1969). [K. L. Nagy *"State vector spaces with indefinite metric in quantum field theory"*, Budapest: Académiai Kiadó (1966).]

[19] R. Haag *"Lectures in theoretical physics"*, Sommer Institute for Th. Phys., Boulder, New York (1961). (cited from [18])