On quantization of electromagnetic field.
IV. Theory of field representations.

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Abstract

We introduce a notion of induced symplectic representation of the Poincare group. Classical relativistic fields are considered as such representations. We describe the method of investigation of these fields in the sense of their reducibility. We introduce the notion of the field oscillator as an inducing Hamiltonian system.

1. Symplectic representations. By Poincare group we will call a connected continuous ten-parameter group of transformations of space-time including both homogeneous transformations (the Lorentz group) and space-time shifts. We will denote the Poincare group by the symbol $\mathcal{P}$.

So far as in these papers we study fields that are relativistic invariant, every solution of equations of the motion is transformed to a solution by a transformation from the Poincare group. So, this defines the action of the Poincare group on the invariant phase space Z .

In a more concrete way this action can be described in the following way. Let we have a transformation $g \in \mathcal{P}$. It acts on points of space-time in the following way:

$$x \;\rightarrow\; x' = gx \;.$$

In the space, where the values of the field function φ_i lie, the transformation g acts as:

$$\varphi_i \;\rightarrow\; \varphi_i' = \Lambda_{ij}\,\varphi_j \;.$$

The action of g in the space Z can be described by the formulas:

$$\underline{c} \longrightarrow \underline{gc}\;, \qquad \varphi_i(x)\,\underline{gc} = \Lambda_{ij}\,\varphi_j(g^{-1}x)\,\underline{c}\;. \tag{1}$$

As we have said in the paper [I], we will consider here only linear fields. The action of the Poincare group $\mathcal{P}$ obviously preserve the linear structure in the space Z . So, in the space Z we have a linear representation of the Poincare group.

Furthermore, the Poincare group preserves in the space Z the symplectic structure. So, we come to linear *symplectic* representations of the Poincare group $\mathcal{P}$. The connection of symplectic representations with unitary will become clear after we define the operation of quantization in the paper [VI]. It will be shown there that in the field theory symplectic representations apparently play a role that is not less important than that of unitary. Anyway, they are connected with the construction of quantized field more closely.

2. "Reduction" of field representation. Consider now the question of reducibility of field representations of the Poincare group $\mathcal{P}$. We will always imply here reducibility in the complex sense. In order to not complexify the space Z let us consider the conjugate space $Z^*_{\mathbb{C}}$. In $Z^*_{\mathbb{C}}$, obviously, acts the conjugate representation of the group $\mathcal{P}$. In order to define this action we just need to read the formula (1) a little differently:

$$\varphi_i(x)\,\underline{\cdot} \longrightarrow \big(g\,\varphi_i(x)\big)\,\underline{\cdot}\;, \qquad \big(g\,\varphi_i(x)\big)\,\underline{c} = \Lambda_{ij}\,\varphi_j(g^{-1}x)\,\underline{c}\;, \tag{2}$$

i. e. we connect the symbol g not with an element of Z , but with an element of $Z^*_{\mathbb{C}}$. Below by a field representation we will always imply the action of the Poincare group by formula (2) in the complexified conjugate space $Z^*_{\mathbb{C}}$.

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For infinite-dimensional representations the notion of reduction requires specification. Now we will explain the structure of field representations under consideration and simultaneously specify in what sense we will talk about their reducibility.

In the case of unitary representations the main tool of investigation is Mackey theorem (see, for example, [1]). This theorem brings investigation of a representation of the Poincare group $\mathcal{P}$ to investigation of the corresponding little Lorentz group $\mathcal{L}_k$. The construction of Mackey induction can be generalized to any linear representations. But a general linear representation (including symplectic) of the group $\mathcal{P}$ can be not induced from the little group. Nevertheless, representations induced in Mackey sense constitute a very wide class, and in fact, we will restrict ourself with them.

Let us make a Fourier transformation of the field and write the result in the form:

$$\widetilde{\varphi}_i(k) = 2\pi\,\delta(k^2 - m^2)\cdot a_i(k)\ . \tag{3}$$

It is natural to call the scalar m a mass of the field, like in quantum theory.

On the grounds of the formula (3) it is natural to say that the field representation decomposes into the direct sum of positive- and negative-frequency subrepresentations:

$$Z^*_{\mathbb{C}} = Z^{*\,(+)}_{\mathbb{C}} \oplus Z^{*\,(-)}_{\mathbb{C}}\ .$$

These two representations we will also call field representations. The projectors to the positive- and negative-frequency subspaces in the Fourier representation are just operators of multiplication by the functions $\theta(+k)$ and $\theta(-k)$, correspondingly.

So far as we consider here real fields, $a_i(-k) = a_i^*(k)$. So, it is natural to suppose that both subspaces $Z^{*\,(+)}_{\mathbb{C}}$ and $Z^{*\,(-)}_{\mathbb{C}}$ are reducible in the same degree. In this connection we can restrict ourself with consideration of the positive-frequency subspace.

Consider now a fixed vector $k^{(0)}$ on the mass surface: $(k^{(0)})^2 = m^2$. And consider the subgroup of the Lorentz group that leaves this vector unchanged, i. e. the so called little group of this vector. Let us denote this group $\mathcal{L}_{k^{(0)}}$. Obviously, the set of values $a_i(k^{(0)})$ corresponding to different values of the index i is transformed linearly under action of transformations from $\mathcal{L}_{k^{(0)}}$. This complex representation of the little group we will suppose to be finite-dimensional. Like in the unitary case, we will say that the field representation under consideration is induced by this representation of the little group.

Let us, in analogy with the unitary case, suppose that the further reducibility of the field representation (i. e. reduction of its positive-frequency part) is fully defined by reducibility of the inducing representation.

Let us note here also that in the case of massive fields, i. e. when $m > 0$, the little group $\mathcal{L}_{k^{(0)}}$ is the group of three-dimensional rotations $SO(3)$. This group is compact, and therefore, its representation can be made unitary by introducing a proper scalar product. Therefore, the inducing representation turns out to be fully reducible, and the equivalence classes of irreducible components, like in the unitary case, are defined by a one integer or half-integer number. It is natural to call this number, like in the quantum case, a *spin* of the irreducible component under consideration.

In the case of zero-mass field the little group, as we know, is the group of motions of a Euclidian plane $E(2)$. This group is not compact, and situation becomes much more complicated than in the unitary case. We will see it soon with example of electromagnetic field.

3. Field oscillator as inducing Hamiltonian system. It is easy to notice that there is a great similarity in description of the harmonic oscillator and the scalar field. Now we will research this phenomenon in detail.

Consider any real field that is written in the Fourier representation as (3). Let us suppose that the Poisson brackets of the field values in the Fourier representation have the form:

$$\{\,\widetilde{\varphi}_i(k)\,,\,\widetilde{\varphi}_j(k')\,\} = B_{ij}(k)\cdot\Big[\,i\,\varepsilon(k)\cdot 2\pi\,\delta(k^2 - m^2)\Big]\cdot(2\pi)^4\delta(k + k')\ . \tag{4}$$

Here $B_{ij}(k)$ is some tensor function. In the cases of the scalar field and the non-physical electromagnetic field, in accordance with the formulas of the paper [I], this function is just a constant. But in general case it is not so. So, let us consider its properties with more details.

First, so far as $B_{ij}(k)$ is multiplied by $2\pi\,\delta(k^2 - m^2)$, we can admit that it is defined only on the mass surface $k^2 = m^2$. Second, using antisymmetry of the Poisson bracket, we get:

$$B_{ij}(k) = B_{ji}(-k)\ . \tag{5}$$

Furthermore, requirements of relativistic invariance make very hard restrictions for $B_{ij}(k)$. Let us fix some point $k^{(0)}$ on the mass surface. If we know the value of the function $B_{ij}(k)$ in this point, then, using relativistic invariance, we can define it in the other points. The value $B_{ij}(k^{(0)})$ is not arbitrary also: it must be invariant under action of the little group $\mathcal{L}_{k^{(0)}}$.

The fact that the Poisson bracket is a complexified real bracket makes another restriction for $B_{ij}(k)$: $B^*_{ij}(k) = B_{ij}(-k)$. Taking into account (5) we can write it also as $B^*_{ij}(k) = B_{ji}(k)$.

Consider now the values $a_i(+k^{(0)})$ and $a_i(-k^{(0)})$. As it was shown in the section 2, they form a representation of the little group $\mathcal{L}_{k^{(0)}}$. It is obvious also that these values form a representation of the group of shifts[1]. A shift by 4-vector l can be described by the formulas:

$$a_i(k^{(0)}) \,\to\, a_i(k^{(0)}) \cdot e^{+ik^{(0)}l} \;, \qquad a_i(-k^{(0)}) \,\to\, a_i(-k^{(0)}) \cdot e^{-ik^{(0)}l} \;.$$

Let us introduce a notation $\tau = k^{(0)}l$. Then we can think that the values $a_i(+k^{(0)})$ and $a_i(-k^{(0)})$ form a representation of the group $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$, where $\mathbb{R}$ is the additive group of real numbers, parameterized by the number τ .

Let us consider now the values $a_i(+k^{(0)})$ and $a_i(-k^{(0)})$ as dynamical variables of some new system. We will call this system a *field oscillator*. In order to economize notations we will write variables of the field oscillator with the same symbols, but we will write as an argument not the vector $k^{(0)}$ but a real number ω ; and $\omega = +1$ corresponds to $k = +k^{(0)}$, and $\omega = -1$ corresponds to $k = -k^{(0)}$. The phase space of the field oscillator can be constructed by realification of the space of inducing representation. The Poisson bracket can be defined by the formulas:

$$\{\, a_i(+1) \,,\, a_j(+1) \,\} = \{\, a_i(-1) \,,\, a_j(-1) \,\} = 0 \;,$$
$$\{\, a_i(+1) \,,\, a_j(-1) \,\} = \, iB_{ij}(+1) \;.$$

Assuming that the Poisson bracket is not degenerate we can, as usually, calculate the symplectic structure.

The action of the group $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$ on the phase space of the field oscillator leaves the Poisson brackets invariant. Therefore it is symplectic.

Now we can interpret the action of a transformation from $\mathbb{R}$, characterized by the parameter τ , as a time shift by the time period τ (the "time" of the field oscillator is a non-dimensional parameter).

We can even define "coordinates" of the field oscillator at "time" t :

$$\varphi_i(t) = \tfrac{1}{\sqrt{2}} \left(\, a_i(+1) \cdot e^{-it} + a_i(-1) \cdot e^{+it} \,\right) \,.$$

We see that the oscillator of a real field has real coordinates.

If we consider the little group $\mathcal{L}_{k^{(0)}}$ as an abstract group, then it does not depend on the choice of the vector $k^{(0)}$. If we describe the oscillator by internal notions of the Hamiltonian formalism (phase space, symplectic structure, symplectic action of the group $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$), then its construction does not depend on the choice of the vector $k^{(0)}$ also. So, for any $\mathcal{P}$ -invariant field we can uniquely juxtapose $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$ - invariant field oscillator.

4. Scalar field representation. Let us apply now the described scheme to the scalar field. The inducing representation in this case is the one-dimensional identity representation. The field oscillator is just the usual real oscillator with one degree of freedom.

So, with respect to the action of the Poincare group, the space $Z^*_{\mathbb{C}}$ decomposes into the direct sum of the positive- and the negative-frequency reducing subspaces, and each of these subspaces is irreducible. These two subspaces we will denote $Z^{*\,(+)}_{\mathbb{C}}$ and $Z^{*\,(-)}_{\mathbb{C}}$. The projections of elements of the space $Z^*_{\mathbb{C}}$ to these two spaces we will provide with labels ${}^{(+)}$ or ${}^{(-)}$, correspondingly:

$$\widetilde{\varphi}^{\,(\pm)}(k) = \theta(\pm k)\,\widetilde{\varphi}(k) \;, \qquad \varphi^{\,(\pm)}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \widetilde{\varphi}^{\,(\pm)}(k) \;.$$

Using formulas from the paper [I], we can calculate the Poisson brackets of these projections:

$$\{\, \widetilde{\varphi}^{\,(\pm)}(k) \,,\, \widetilde{\varphi}^{\,(\pm)}(k') \,\} = 0 \;,$$

[1] Speaking more generally, these values form a representation of the seven-parameter subgroup of the Poincare group that includes the little group $\mathcal{L}_{k^{(0)}}$ and shifts.

$$\{\,\widetilde{\varphi}^{\,(\pm)}(k)\,,\,\widetilde{\varphi}^{\,(\mp)}(k')\,\} = -\Big[\,i\,\theta(\pm k)\,\varepsilon(k)\cdot 2\pi\,\delta(k^2-m^2)\Big]\cdot(2\pi)^4\delta(k+k')\ .$$

In the coordinate representation we have, correspondingly:

$$\{\,\varphi^{\,(\pm)}(x)\,,\,\varphi^{\,(\pm)}(x')\,\} = 0\ ,$$

$$\{\,\varphi^{\,(\pm)}(x)\,,\,\varphi^{\,(\mp)}(x')\,\} = -D_m^{(\pm)}(x-x')\ .$$

Here symbols $D_m^{(+)}(y)$ and $D_m^{(-)}(y)$ denote the positive and the negative-frequency parts of the function $D_m(y)$:

$$D_m^{(\pm)}(y) = \int \frac{d^4k}{(2\pi)^4}\,e^{-iky}\cdot\Big[\,i\,\theta(\pm k)\,\varepsilon(k)\cdot 2\pi\,\delta(k^2-m^2)\Big]\ . \qquad (6)$$

The given formulas turn out to be very useful for calculation of propagators of the quantized field. But they naturally follow from the classical theory.

5. Electromagnetic field representation. According to the section 2, the representation of the non-physical electromagnetic field decomposes also into the direct sum of positive- and negative-frequency representations. The formulas of the section 4 can be rewritten in the following way:

$$\widetilde{A}_\mu^{(\pm)}(k) = \theta(\pm k)\,\widetilde{A}_\mu(k)\ , \qquad A_\mu^{(\pm)}(x) = \int \frac{d^4k}{(2\pi)^4}\,e^{-ikx}\,\widetilde{A}_\mu^{(\pm)}(k)\ ,$$

$$\{\,\widetilde{A}_\mu^{(\pm)}(k)\,,\,\widetilde{A}_\nu^{(\pm)}(k')\,\} = 0\ ,$$

$$\{\,\widetilde{A}_\mu^{(\pm)}(k)\,,\,\widetilde{A}_\nu^{(\mp)}(k')\,\} = g_{\mu\nu}\cdot\Big[\,i\,\theta(\pm k)\,\varepsilon(k)\cdot 2\pi\,\delta(k^2)\Big]\cdot(2\pi)^4\delta(k+k')\ ,$$

$$\{\,A_\mu^{(\pm)}(x)\,,\,A_\nu^{(\pm)}(x')\,\} = 0\ ,$$

$$\{\,A_\mu^{(\pm)}(x)\,,\,A_\nu^{(\mp)}(x')\,\} = g_{\mu\nu}\,D_0^{(\pm)}(x-x')\ .$$

Here $D_0^{(\pm)}(y)$ are functions (6) with $m=0$. In this case they can be written also as:

$$D_0^{(\pm)}(y) = \frac{1}{i\,(2\pi)^2}\cdot\frac{1}{\pm\,y^2 - i\,0\cdot\varepsilon(y)} = \frac{1}{4\pi}\,\varepsilon(y)\,\delta(y^2) \pm \frac{1}{i\,(2\pi)^2}\,\mathcal{P}\,\frac{1}{y^2}\ .$$

The inducing representation in this case is the complex vector representation of the little group $\mathcal{L}_{k^{(0)}}$ with $(k^{(0)})^2 = 0$. A more detailed description of this representation is given in the paper [V]. It has two composition series:

$$\{0\} = M_{\mathbb{C}}^0 \subset M_{\mathbb{C}}^{\parallel} \subset M_{\mathbb{C}}^{(+1)} \subset M_{\mathbb{C}}^{\perp} \subset M_{\mathbb{C}}^4 = M_{\mathbb{C}}\ ,$$

$$\{0\} = M_{\mathbb{C}}^0 \subset M_{\mathbb{C}}^{\parallel} \subset M_{\mathbb{C}}^{(-1)} \subset M_{\mathbb{C}}^{\perp} \subset M_{\mathbb{C}}^4 = M_{\mathbb{C}}\ .$$

Under induction these two composition series turn into composition series of the field representation.

$$\{0\} = Z_{\mathbb{C}}^{*\,(+)\,0} \subset Z_{\mathbb{C}}^{*\,(+)\,\parallel} \subset Z_{\mathbb{C}}^{*\,(+)\,(+1)} \subset Z_{\mathbb{C}}^{*\,(+)\,\perp} \subset Z_{\mathbb{C}}^{*\,(+)\,4} = Z_{\mathbb{C}}^{*\,(+)}\ ,$$

$$\{0\} = Z_{\mathbb{C}}^{*\,(+)\,0} \subset Z_{\mathbb{C}}^{*\,(+)\,\parallel} \subset Z_{\mathbb{C}}^{*\,(+)\,(-1)} \subset Z_{\mathbb{C}}^{*\,(+)\,\perp} \subset Z_{\mathbb{C}}^{*\,(+)\,4} = Z_{\mathbb{C}}^{*\,(+)}\ .$$

Here the notations for the subspaces are in natural concordance with the notations of the paper [I].

The field oscillator in this case turns out to be a system with eight-dimensional phase space.

In connection with the division of the field into the positive- and the negative-frequency parts, let us notice also that so far as electromagnetic field is real, the Lorentz condition for scattered states can be written in three equivalent forms:

$$-\,i\,k_\mu\,a_\mu(k)\,\overline{\mathrm{rad}} = 0\ ,$$

$$-\,i\,k_\mu\,a_\mu^{(+)}(k)\,\overline{\mathrm{rad}} = 0\ ,$$

$$-\,i\,k_\mu\,a_\mu^{(-)}(k)\,\overline{\mathrm{rad}} = 0\ .$$

So far as positive- and negative-frequency parts of the field play different roles in quantization, in the paper [VI] we will see that in quantum theory there is an analog of only the second form of this condition.

References

[1] G. W. Mackey *„Predstavleniya grupp v gilbertovom prostranstve“*, appendix in the book [2]. [G. W. Mackey "Group representations in Hilbert space", appendix in the book [2].]

[2] I. E. Segal *„Matematicheskie problemy relyativistskoy fiziki“*, M.: Mir (1968). [I. E. Segal *"Mathematical problems of relativistic physics"*, Providence, Rhode island: AMS (1963).]