The certainty principle II
Proof of the uncertainty principle

D. A. Arbatsky*

March, 2006

Abstract

A more detailed derivation of the Heisenberg uncertainty principle from the certainty principle is given.

Introductory remarks. After publication of the paper [1] I received many letters, including those with references. In this connection, I think it is necessary to specify the following:

1. The metric introduced by me ("quantum angle") is known to mathematicians from 1904 as *Fubini-Study metric*.
2. The correct "uncertainty relation" (in fact, *certainty* relation) for the quantities energy - time was first suggested by Mandelshtam and Tamm [2].
3. Mandelshtam and Tamm studied a quantum system in the Schrödinger representation and did not use group theory methods. For this reason they could not understand that their result has more general character.
4. In contrast, I used group theory formulations. And implied that the system can be studied, in particular, in the representation of relativistic canonical quantization [4]. This allowed me to formulate the certainty principle and to suggest more general inequalities.

Some of my critics refused to believe that the Heisenberg uncertainty principle is really a consequence of the certainty principle. They claimed that "this can not be so, because this can never be so".

Nevertheless, this is so. And here I give a more detailed explanation.

Derivation of the uncertainty principle. Suppose that for a given quantum system we succeeded to find some observable X , that can be considered in some sense a "coordinate operator".

Suppose that X is a self-adjoint operator with continuous spectrum, $X = X^*$. Let us denote $\Omega_{(a,b)}$ its spectral projector[1] for an arbitrary real interval (a,b) .

Suppose also that we have a self-adjoint operator P , $P = P^*$, such that for any a , b and δx we have equality:

$$e^{+\,i\,\delta x\,P/\hbar}\,\Omega_{(a+\delta x,b+\delta x)}\,e^{-\,i\,\delta x\,P/\hbar} \;=\; \Omega_{(a,b)} \;,$$

i. e. P is a "generator of spectral shifts" for X . As we know, such an operator is usually an operator of momentum.

Suppose now that the system is in state $\rangle$, $\langle\,|\,\rangle = 1$.

The quantity $\langle\,\Omega_{(a,b)}\,\rangle$, obviously, defines the probability to find the system inside the interval (a,b) . Let us define such l and r , that

$$\langle\,\Omega_{(-\infty,l)}\,\rangle \;=\; \langle\,\Omega_{(r,+\infty)}\,\rangle \;=\; \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\cos^2 1} \;\approx\; 0,07926\ldots$$

*http://daarb.narod.ru/ , http://wave.front.ru/

[1] Roughly speaking, spectral projector is an operator nullifying wave function in X -representation outside of the given interval.

It is easy to see, that l and r exist[2]. So, the quantity $\delta_{\rangle} X = r - l$ can be naturally called "uncertainty" of the coordinate X .

T h e o r e m. *The following inequality takes place (the uncertainty principle):*

$$\boxed{\delta_{\rangle} X \Delta_{\rangle} P \geqslant \hbar} \tag{1}$$

In order to prove this theorem let us first estimate the scalar product of the vector $\rangle$ and the shifted vector $e^{-i\,\delta_{\rangle} X\,P/\hbar}\rangle$:

$$\begin{aligned}
\big|\langle\,|\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| &= \big|\langle\,(\,\Omega_{(-\infty,r)}+\Omega_{(r,+\infty)}\,)\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| = \\
&= \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle + \langle\Omega_{(r,+\infty)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| \leqslant \\
&\leqslant \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| + \big|\langle\Omega_{(r,+\infty)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| = \\
&= \big|\langle\,(\Omega_{(-\infty,r)})^2\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| + \big|\langle(\Omega_{(r,+\infty)})^2\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\big| = \\
&= \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\,\Omega_{(-\infty,l)}\rangle\big| + \big|\langle\Omega_{(r,+\infty)}\,e^{-i\,\delta_{\rangle}X\,P/\hbar}\,\Omega_{(l,+\infty)}\rangle\big| \leqslant \\
&\leqslant \langle\,\Omega_{(-\infty,r)}\rangle^{1/2}\langle\,\Omega_{(-\infty,l)}\rangle^{1/2} + \langle\,\Omega_{(r,+\infty)}\rangle^{1/2}\langle\,\Omega_{(l,+\infty)}\rangle^{1/2} = \\
&= \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-\cos^2 1}}\cdot\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\cos^2 1}} + \\
&\qquad + \sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\cos^2 1}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-\cos^2 1}} = \cos 1 \ .
\end{aligned}$$

Here, coming from the fifth to the sixth line, we estimated both terms by the Cauchy-Bunyakovsky-Schwarz inequality.

So, for the quantum angle between the initial and the shifted vectors we have estimation:

$$\angle\big(\,\rangle\ ,\ e^{-i\,\delta_{\rangle}X\,P/\hbar}\rangle\,\big) \ \geqslant\ 1\ .$$

But it means that under the action of $e^{-i\,\delta_{\rangle}X\,P/\hbar}$ the vector $\rangle$ changes *substantially* [1].

Applying the *certainty* principle [1], we directly get (1). ■

Discussion. Historically, it stacked up so [5] that now in all textbooks the Heisenberg uncertainty principle is illustrated with the help of the Kennard inequality,

$$\Delta_{\rangle} X\,\Delta_{\rangle} P \ \geqslant\ \frac{\hbar}{2}\ \ ,$$

which is not identical with (1). Its undoubted virtue is that its proof is easier to understand for a person, who just starts to study quantum mechanics. Nevertheless, I believe that (1) is more fundamental.

Acknowledgements. In closing I want to thank T. A. Bolokhov, A. V. Ossipov, E. V. Aksyonova, A. Yu. Toschevikova, A. Kleyn, M. Gatti, A. K. Pati, and P. Enders for helpful discussions, references and help in procuring literature.

References

[1] D. A. Arbatsky "The certainty principle" (2005),
http://daarb.narod.ru/tcp-eng.html , http://wave.front.ru/tcp-eng.html , arXiv:quant-ph/0506165 .
[Rus.: D. A. Arbatsky „Printsip opredelyonnosti" (2005),
http://daarb.narod.ru/tcp-rus.html , http://wave.front.ru/tcp-rus.html , arXiv:quant-ph/0506165 .]

[2] But, generally speaking, they are not unique. In order to eliminate this non-uniqueness, it is convenient to choose l maximum of the possible, and r — minimum. Then the distance $r - l$ will be minimum.

[2] L. I. Mandelshtam, I. E. Tamm, “The uncertainty relation between energy and time in nonrelativistic quantum mechanics”, J. Phys. (USSR) **9**, 249-254 (1945).
[Rus.: L. I. Mandelshtam, I. E. Tamm „Sootnoshenie neopredelyonnosti energiya-vremya v nerelyativistskoy kvantovoy mekhanike“, Izv. Akad. Nauk SSSR (ser. fiz.) **9**, 122-128 (1945), in [3], p. 306-315.]

[3] L. I. Mandelshtam, *Polnoe sobranie trudov*, v. 2, M.-L. (1947).

[4] D. A. Arbatsky “What is “relativistic canonical quantization”?” (2005),
http://daarb.narod.ru/wircq-eng.html , http://wave.front.ru/wircq-eng.html , arXiv:math-ph/0503029 .
[Rus.: D. A. Arbatsky „Chto takoe „relyativistskoe kanonicheskoe kvantovanie“?“ (2005),
http://daarb.narod.ru/wircq-rus.html , http://wave.front.ru/wircq-rus.html , arXiv:math-ph/0503029 .]

[5] J. Hilgevoord, J. Uffink “The uncertainty principle”, in *The Stanford encyclopedia of philosophy*, ed. E. N. Zalta (2001), http://plato.stanford.edu/archives/win2001/entries/qt-uncertainty/ .