What is “Relativistic Canonical Quantization”?

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January, 2005

Abstract

The purpose of this review is to give the most popular description of the scheme of quantization of relativistic fields that was named *relativistic canonical quantization* (RCQ). I do not give here the full exact account of this scheme. But with the help of this review any physicist, even not a specialist in the relativistic quantum theory, will be able to get a general view of the content of RCQ, of its connection with other known approaches, of its novelty, and of its fruitfulness.

Invariant Hamiltonian formalism

It is generally known that for construction of quantized fields it appears to be useful to calculate Poisson brackets of field values. And a Poisson bracket is the notion of Hamiltonian formalism.

If we have a problem to create a completely relativistic-invariant scheme of quantization of fields, then we wish, first of all, to formulate Hamiltonian formalism in relativistic-invariant form.

Science solved this problem for surprisingly long time. In fact, it was generally accepted that Hamiltonian formalism can not be formulated in explicitly relativistic-invariant form.

Nevertheless, with development of methods of symplectic geometry, it became possible to formulate Hamiltonian formalism on the basis of such notions that turned out to have relativistic-invariant analogs. Here these notions are: phase space, symplectic structure on phase space, canonical action of the one-parameter group of time shifts in phase space.

For relativistic fields the analogs of these notions are, correspondingly: invariant phase space, symplectic structure on invariant phase space, canonical action of the Poincare group on invariant phase space.

Invariant phase space. A point of usual phase space describes the dynamical state of system in the fixed moment of time. If for each initial state of the system the equations of motion are solvable, and uniquely, then we can speak about one-to-one correspondence between the phase space of the system (for the fixed moment of time) and the set of solutions of the equations of motion. This set of solutions of the equations of motion is called *invariant phase space.*

Invariant phase space possesses natural structure of manifold. If dynamical system is a relativistic field, then as possible coordinate functions on invariant phase space we can use values of magnitude of the field in fixed points of space-time.

So, from the viewpoint of structure of invariant phase space, possible values of field in a fixed point of space-time are just one of the huge set of functions on invariant phase space.

Symplectic structure. It can seem that the absence of any special coordinate functions on invariant phase space, that could be used to identify points of this space, makes this space an empty abstraction.

But this is not so. Because it is possible to show that invariant phase space, like usual one, possesses symplectic structure.

When we state one-to-one correspondence of points of the invariant phase space and points of a usual phase space, taken in a fixed moment of time, it turns out that their symplectic structures correspond to each other. From this

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it is clear that existence of the symplectic structure on invariant phase space is a fact which is a *generalization* of Liouville and Poincare theorems.

As regards Poisson brackets, they are defined through symplectic structure. Their mathematical definition, in fact, does not change. But so far as this definition is applied to objects of other nature, it turns out that such Poisson brackets are a deep generalization of the usual. For example, they can be calculated between values of field in *different* moments of time.

Action of Poincare group. If the initial Lagrangian of a field is relativistic-invariant, then the equations of motion are also relativistic-invariant. So, under any transformation from the Poincare group a solution of the equations of motion is transformed into solution.

In other words, the Poincare group *acts* on invariant phase space.

So far as the symplectic structure on invariant phase space is defined through Lagrangian, it turns out to be invariant under the action of the Poincare group. So, the Poincare group acts on the invariant phase space canonically.

Field representations. For the problem of quantization we can restrict ourself with linear fields. These fields are characterized by the property that linear combination of solutions of equations of motion is a solution. From the point of view of invariant phase space, we can say that in this case it possesses natural linear structure.

The Poincare group preserves this linear structure. So, the Poincare group acts on invariant phase space as a group of *symplectic* transformations.

So far as for the purpose of quantization it is necessary to analyse this action of the Poincare group by methods of group theory, it is useful to use the following terminology. We say that linear relativistic fields define *symplectic representations* of the Poincare group. These representations (and also their conjugate and complexified) are also called *field representations*.

We will not become more profound here in the theory of field representations. Let us just notice that this theory is quite analogous to the Wigner-Mackey theory of unitary representations of Poincare group, but it plays more fundamental role for the theory of quantized fields.

Construction of quantized field

So, now we already have all necessary structures that are used for *construction* of quantized field. Here they are: invariant phase space of linear classical field, symplectic structure on this space, properly classified invariant (with respect to the Poincare group) subspaces of field representation, and the proper set of classical field values that are "quantized".

In this review I omit a description of exact construction of quantized field. From the point of view of an algebraist all methods used for it are well known. Similar methods are used for construction of universal enveloping algebras for Lie algebras, for construction of Grassmanian algebras etc.

Let us give now a list of some most important properties of the quantization under consideration.

- Quantization is performed *constructively*. Properties of quantized field (for example, commutation relations) are not postulated, but follow from the construction.
- The construction is explicitly relativistic-invariant.
- The method makes possible to see, how quantum system acquires integrals of motion connected with symmetry group, like initial classical system. So, it becomes possible to formulate the mathematically rigorous quantum analog of the Noether theorem.
- The method also allows to see anew the origin of discrete symmetries, including *anti-unitary* transformation of time reversal.
- The quantization scheme naturally covers the possibility of quantization in a space with indefinite scalar product.

Application to electromagnetic field

The problem of quantization of electromagnetic field was one of the main stimuli for creation of RCQ.

It is known for a long time, that for the problems of quantum theory it is necessary to describe electromagnetic field by means of vector potential. It is also known that an attempt of quantization of such a vector field leads to the necessity of consideration of indefinite scalar product in quantum space of states.

An indefinite scalar product, in contrast to positive-definite case, does not define topology.

This problem was just ignored up to now (in educational literature). By analogy with some other fields, it was believed that the space of states of quantized electromagnetic field *must* be Hilbert space, at least from topological point of view.

The RCQ method has shown that this *is not so*.

Other applications

The RCQ method, of course, can be applied to other fields (for example, to scalar, to electron-positron etc.)

It is known that, for example, the scalar field has only one quantization in Hilbert space. Such a quantization is already constructed, of course. And the RCQ method (if we restrict ourself with positive-definite scalar product) *can not* lead to other quantization.

Of course, the RCQ method leads in this case to equivalent quantization. But the undoubted merit of the method is that the construction is performed in the frame of the general scheme, without any “guesses”, “classical analogies” etc.

It happens so, because the RCQ method is a *rigorous mathematical construction*.

References

[1] D. A. Arbatsky “On quantization of electromagnetic field” (2002), http://daarb.narod.ru/qed-eng.html , http://wave.front.ru/qed-eng.html , arXiv:math-ph/0402003 . [Rus.: D. A. Arbatsky „O kvantovanii elektromagnitnogo polya“ (2002), http://daarb.narod.ru/qed-rus.html , http://wave.front.ru/qed-rus.html , arXiv:math-ph/0402003 .]